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AFDELING ZUIVERE WISKUNDE ZW 1/71 JANUARY

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PROBLEMS IN THE TOPOLOGY OF INFINITE-DIMENSIONAL SPACES AND MANIFOLDS

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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The ZW-reports are research publications, written at the Department of Pure Mathematics of the Mathematical Centre in Amsterdam. They are reviewed in Mathematical Reviews and the Zentralblatt für Mathematik and they are sent to about 60 universities and libraries in several countries in exchange for their series.

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Problems in the Topology of Inifinite-Dimensional Spaces and Manifolds

This problem list is derived from earlier problem lists prepared at conferences in Ithaca (January 1969) and Baton Rouge (December 1969), and from problems suggested at the conference in Oberwolfach (September 1970). This list has been prepared at the Mathematisch Centrum in Amsterdam. It is not a complete list of all current open questions known to the various participants, but it does include (rephrasings of) most or all open questions of the earlier problem lists and of the Anderson-Bing paper, BAMS 74 (1968), 771-792. In general, naturally arising questions concerning generalizations of known procedures and theorems to other spaces are not included. It is likely that a few of the problems listed here are inadequately worded, are trivial, or are known.

The following mathematicians are possible sources of information including later status of problems in categories LS, Q, QM, PF and M: R.D. Anderson, C. Bessaga, T.A. Chapman, D.W. Henderson, R.M. Schori, James E. West and Raymond Wong. Similarly, D. Burghelea, J. Eells, D. Elworthy and N. Kuiper are possible sources of information on problems in categories M, LG and D.

NOTATION

(a) $\ell^2 = \ell_2 = H$ = separable Hilbert space.

(b) $I^\infty = Q$ = Hilbert cube = $\Pi_{i>0} I_i$, where $I_i = [-1,1]$.

(c) $s = \Pi_{i>0} I_i^0$, where $I_i^0 = (-1,1)$, = countable infinite product of lines.

(d) "$\simeq$" = "has the same hopotopy type as".

(e) "$\cong$" = "is homeomorphic to".

(f) "$\overset{\sim}{\equiv}$" = "is (C^∞-) diffeomorphic to".

(g) "E" or "F" = topological vector space (TVS) (further conditions are frequently specified).

(h) "E-manifold" = Paracompact Hausdorff manifold modeled on E, i.e. admitting an open cover by sets homeomorphic to E.

(i) "Q-manifold" = locally compact metric space which has an open covering by sets homeomorphic to open subsets of $Q = I^\infty$.

(j) A closed subset A of a space X is a Z-set in X (or has Property Z in X) if for each non-empty contractible open set U in X, $U \setminus A$ is non-empty and contractible. Note: for most of the spaces concerned, contractibility may be replaced by homotopy triviality. Various alternative forms are known, e.g. see Torunczyk, "Remarks on Anderson's paper on topological infinite deficiency".
Z-sets have played an important and useful role in many of the techniques and problems of infinite-dimensional point-set topology.

(k) For X = Q, s or E we let X^n be the n-fold product of X by itself and X^ω or X^∞ be the countable infinite product of X by itself.
For X an infinite coordinate space, let
$X_f = \{(x_i) \in X \mid \text{for all but finitely many } i, x_i = 0\}$.
Let $(X^\omega)_f$ be denoted by X_f^ω.

(l) A topological vector space F is said to satisfy
Property A_1 if $F \cong F^\omega$ and
Property A_2 if $F \cong F_f^\omega$.

These properties are particularly applicable to classes of spaces which include non-separable spaces. They give a certain countable product structure to such spaces and make them more like $s \cong s^{\omega}$ and $s_f \cong s_f^{\omega}$.

(m) A subset M of a metric space X is said to have the (<u>finite-dimensional</u>) <u>compact absorption property</u> [(f-d) cap] if $M = \cup_{i>0} M_i$ such that

(i) for each $i > 0$, M_i is a (finite-dimensional) compactum

(ii) $M_i \subset M_{i+1}$,

(iii) for any (finite-dimensional) compactum $K \subset X$, any open cover $\mathcal{U}$ of X and any positive integer m, there exists an integer $n > 0$ and a homeomorphism g of X onto X such that $g(K) \subset M_n$, $g \mid K \cap M_m = \mathrm{id}$, and g is <u>limited by</u> $\mathcal{U}$ (i.e. for any $x \in X$, there exists a $U \in \mathcal{U}$ such that $x, g(x) \in U$).

It is understood that finite-dimensionality is used in the three places as indicated or not at all. Bessaga and Pelczynski have introduced a similar concept using skeletonized sets and estimated extensions. See also Torunczyk in this regard. West has introduced a non-separable version of the f-d cap.

The following is hopefully a representative list of types of problems that have played an important role in the recent development of infinite-dimensional topology. Below is a chart giving a brief summary of our understanding of the status of these various questions. Let X represent either Q, H, or F. Consider columns I, II, and III to represent spaces in the topological category and column IV to represent spaces in the differential and/or linear topological category. In Column III we assume that F satisfies either A_1 or A_2. In some instances a more general form of A_1 or A_2 can be used. The full specific conditions for the various results listed in Column III should be checked by the reader.

1) Is the space of automorphisms of X (with the appropriate topology) contractible? In the topological category, automorphism $\equiv$ homeomorphism; and in the linear topological category, automorphism $\equiv$ linear map with continuous inverse (i.e. the general linear group of the space).

2) Stability of manifolds: That is, if M is a manifold modeled on X, is $M \times X$ homeomorphic (diffeomorphic) to M?

3) If M is a manifold modeled on X, does there exist a homeomorphism (diffeomorphism) of M onto an open subset of X?

4) Let M and N be manifolds modeled on X. If M and N are of the same homotopy type, are they homeomorphic (diffeomorphic)?

5) If K is a closed subset of manifold M modeled on X, then is the property of K being a Z-set equivalent to K being topologically ∞-deficient (i.e. of topological ∞ codimension)? Furthermore, do these conditions imply that K is <u>negligible</u> (i.e. that there exists a homeomorphism h of $M \setminus K$ onto M)? Or that K is <u>strongly negligible</u> (i.e. h is arbitrarily small)?

6) If K_1 and K_2 are Z-sets in the manifold M modeled on X and $h: K_1 \to K_2$ is a homeomorphism which is homotopic in M to the identity map on K_1, then can h be extended to a homeomorphism on M?

7) If M is a manifold modeled on X, is $M \cong |K| \times X$, where K is a complex, and conversely?

8) If M is a manifold modeled on X, do (f-d) cap-sets exist? Furthermore, if two (f-d) cap-sets are given in M, then does there exists a space homeomorphism taking one onto the other? Let A be an (f-d) cap-set in M, let K be a compact Z-set and let L be a countable union of (f-d) compact Z-sets in M. Is $(M,A) \cong (M,A \setminus K) \cong (M,A \cup L)$?

X =	I Q	II ℓ_2 (or separable Fréchet)	III F satisfying A_1, or A_2	IV C^∞-Hilbert
1) Aut (X) contractible.	Wong and Renz	Wong and Renz	Wong and Renz	$GL(\ell_2)$ Kuiper Also ℓ_p, c_0 $GL(c_0 \oplus \ell_2)$ not contractible
2) $M \times X \cong M$	Anderson-Schori	Anderson-Schori	Schori (MTVS)	Kuiper-Burghelea, Eells-Elworthy
3) open $M \subset X$	Chapman (partial results)	Henderson	Henderson-Schori (MTVS)	Eells-Elworthy
4) homotopy equiv. $\equiv$ homeomorphic	Chapman and West (partial results)	Henderson, Kuiper-Burghelea and Moulis	Henderson-Schori (MTVS)	Kuiper-Burghelea, Eells-Elworthy, Moulis
5) K is Z-set in M $\equiv$ K is ∞-def. $\Longrightarrow$ K negligible	Chapman but negligibility is not true	Chapman Z-set $\equiv$ ∞-def. $\left.\begin{matrix}\text{Anderson}\\\text{Henderson}\\\text{West}\end{matrix}\right\} \Rightarrow$ strong neg.	Chapman for A_1 Cutler, partial results	Z-set $\not\Rightarrow$ ∞-def. $M-K^{\text{Z-set}} \cong M$
6) extending homeomorphisms on Z-sets	Anderson-Chapman (with proper homotopies)	Anderson-McCharen	Henderson (appropriate restrictions) Chapman	--
7) $M \cong \lvert K\rvert \times X$ K locally fin. simplicial complex	Chapman (for open subsets of Q) Converse by West	Henderson Converse by West	West (loc. fin. dim. sim. complex)	--
8) (f-d) cap properties	Anderson, Bessaga-Pelczynski and Torunczyk (spaces), Chapman (manifolds)		West (partial)	--

Many partial results for Q-manifolds have been obtained by Chapman. Henderson, Schori, West and others have various generalization to other spaces under III.

LS. Problems about Linear Spaces

LS1. Is every separable normed space homeomorphic to some pre-Hilbert space?

LS2. Let X be a pre-Hilbert space. Is $X \times R \cong X$? $X \times X \cong X$? $X_f^{\omega} \cong X$ or $X^{\omega} \cong X$?

Conjecture: Answer is no for uniform homeomorphisms?

LS3. If a σ-compact separable normed space contains a copy of I^{∞}, is it homeomorphic to $\{x \in \ell_2 : \sum x(n).n < \infty\}$?

LS4. A centered system is a family of sets with the finite intersection property. A linked system is a family of sets such that any two of its members have a non-void intersection. The reals can be defined by the maximal centered systems of (finite unions of) rational intervals. Is the analogous space of maximal linked systems of finite unions of rational intervals homeomorphic to ℓ_2? This space is known to be infinite dimensional, complete metric, contractible, connected, locally connected, and not locally compact where the metric may be defined by considering the rationals on $[0,1]$ and using a natural distance between maximal linked systems.

LS5. Let K_1 and K_2 be closed disjoint subsets of ℓ_2. Is there an element $h \in H(\ell_2)$ such that $d(h(K_1), h(K_2)) > 0$? Remark: The analogous problem for E^n is relatively easy by using the compactness of bounded subsets of $K_1 \cup K_2$ and an exponential type radial expansion of E^n. Henderson has a non-trivial argument for the ℓ_2-case if K_1 is assumed to be a Z-set. Similar problems can be posed for other metric TVS's.

LS6. Is every infinite-dimensional Banach space homeomorphic to some Hilbert space?

LS7. For every infinite-dimensional Banach space E, is $E \cong E^{\omega}$? It is known, by Bessaga and Pelczynski, that $E \cong E^{\omega}$ for every infinite-dimensional Hilbert space.

The following questions deal with special types of homeomorphisms, Lipschitz or uniformly continuous. Let X, Y be metric spaces. A map f of X into Y is <u>Lipschitz</u> if there is a $K > 0$ such that $d(f(x),f(y)) \leq K.d(x,y)$, for all x,y. We say that f is a <u>Lipschitz</u> isomorphism of X onto Y if f is 1-1, onto and both f, f^{-1} are Lipschitz maps.

A map $f: X \to Y$ is <u>uniformly continuous</u> if for each $\epsilon > 0$ there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x),f(y)) < \epsilon$. A homeomorphism f is a <u>uniform homeomorphism</u> if both f and f^{-1} are uniformly continous.

<u>LS8</u>. Let K_1, K_2 be homeomorphic Z-sets in F. For a given homeomorphism h of K_1 onto K_2, does there exist a homeomorphism u of F onto itself such that the induced map $h_* = uhu^{-1}$ of $u(K_1)$ onto $u(K_2)$ is a Lipschitz isomorphism?

Wong has shown that the answer is yes when $F = \ell_p$ for $1 \leq p < \infty$.

<u>LS9</u>. Let K_1, K_2 be Z-sets in ℓ_2 and let f be a Lipschitz isomorphism of K_1 onto K_2. Can f be extended to a Lipschitz isomorphism of ℓ_2 onto itself?

This is known to be true when K_1 is compact.

<u>LS10</u>. Does a homeomorphism h between two compact subsets of F always extend to a uniform homeomorphism H of F onto itself?

<u>LS11</u>. (a) If two Banach spaces are uniformly homeomorphic, are they then isomorphic? (true if one is a Hilbert space).
(b) Is the following subgroup G of the additive group of $L_2[0,1]$ uniformly homeomorphic to $L_2[0,1]$? G consists of all L_2-functions which have integers as values for almost all x in [0,1].

<u>LS12</u>. Is every separable metric space uniformly homeomorphic to some subset of c_0?

<u>LS13</u>. Are the unit balls in c_0 and C[0,1] uniformly homeomorphic?

<u>LS14</u>. What about the concept of "boundary" in uniform topology?
(a) Does there exist a uniform homeomorphism of the closed unit ball in ℓ_2 onto itself such that 0 is mapped to a point on the boundary?
(b) Is a closed half-space of ℓ_2 uniformly homeomorphic to ℓ_2?
(c) Is the closed unit ball in ℓ_2 uniformly homeomorphic to the set $\{x \mid r_1 \leq ||x|| \leq r_2\}$, $r_2 > r_1$?

Q. Problems about the Hilbert cube (See also QM and PF)

Problems 1-5 below are concerned with compactifications of s at the Hilbert cube $Q = I^{\infty}$.

Q1. Let $s \subset N \subset Q$. What are necessary and sufficient conditions that $s \cong N$?

It is obvious that N must be a G_{δ}-subset of Q and it is known that if $Q \setminus N$ contains an f-d cap-set, then $N \cong s$. Is this condition necessary? For the existence of a homeomorphism $h: Q \to Q$ with $h(s) = N$, it is necessary and sufficient that N be a G_{δ}-subset of Q and that $Q \setminus N$ contain a cap-set.

Q2. In Q1 assume that $Q \setminus N$ is a dense (in Q) countable union of disjoint finite-dimensional cubes (or disjoint Hilbert cubes) σ_i with σ_i a cube in an endslice and slightly smaller than the endslice.

Such an N can have the property (or must have the property) that every compact subset of N is a Z-set in N. If it could be shown that every Z-set in N is strongly negligible in N, then $N \cong s$. (See problem M4).

Q3. Let f be a homeomorphism of s onto a dense subset of Q. Is there a map g of Q onto Q such that $g|s$ is a homeomorphism of s onto f(s)? (This is known for $Q \setminus f(s) \cong s_f$). If $Q \setminus f(s)$ is homogeneous, what other conditions guarantee that $Q \setminus f(s)$ is homeomorphic to $Q \setminus s$ or s_f?

Q4. If h is a homeomorphism of s onto itself, is there a homeomorphism $g: s \to s$ such that ghg^{-1} can be extended to a homeomorphism of Q?

Wong has shown that if K is a Z-set in s and h is a homeomorphism of K onto K, then there is a homeomorphism $g: s \to s$ such that $ghg^{-1}|g(K)$ can be extended to a homeomorphism of Q which takes s onto itself.

Q5. Let U be an open subset of s and $f: U \to U$ a map. Are there some reasonable conditions such that f admits a continuous extension $\tilde{f}: \tilde{U} \to \tilde{U}$ to some open subset $\tilde{U}$ of Q with $U = \tilde{U} \cap s$?

Q6. Let Q and Q' be Hilbert cubes with $Q' \subset Q$. Does there exist a Hilbert cube $Q'' \subset Q'$ such that Q'' is a Z-set in Q?

Q7. Let G be an upper semi-continuous decomposition of $X = Q$ (or s, or a separable F or Q-manifold) by compact continua. Let ND(G) be the collection of non-degenerate elements of G. If each element of G has the "shape" of a point (as defined by Borsuk) and if $\mathrm{Cl}(\cup \mathrm{ND}(G))$ is a Z-set, is the hyperspace G^* of G homeomorphic to X? The question is open for ND(G) countable.

Chapman has shown that a Z-set K in I^∞ has the shape of a point iff $I^\infty \setminus K \cong I^\infty \setminus \{\text{point}\}$. Thus such a K is strongly cellular, i.e. it is the intersection of a decreasing sequence of open subsets of I^∞, each one being homeomorphic to $I^\infty \setminus \{\text{point}\}$.

Q8. Are the single point set and Q the only homogeneous contractible compacta?

Q9. Is the space of all closed subsets (with the Hausdorff metric) of an n-cell or Q homeomorphic to Q? This is not known even for a 1-cell.

West has shown that $2^I \times Q \cong 2^I_{0,1} \times Q \cong Q$, where 2^I is the space of closed subsets of I and $2^I_{0,1}$ is the subset of 2^I consisting of all closed subsets of I containing 0 and 1.

Q10. Is the space of all subcontinua of an n-cell $(n > 1)$ or of Q homeomorphic to Q? (It is easily seen that the space of all subcontinua of a 1-cell is a 2-cell).

Q11. Identify classes of subsets of ℓ_2 which are all homeomorphic to Q. The result should be more general than the Keller characterization of all infinite-dimensional compact convex subsets of ℓ_2 as homeomorphic to Q. Such results might be of value in problems 9 or 10 above.

Q12. Let f and g be involutions (i.e. homeomorphisms of period 2) of Q each having exactly one fixed point. Are f and g equivalent, i.e. does there exist a homeomorphism $h: Q \to Q$ for which $h^{-1} \circ g \circ h = f$? More generally what can be said if the fixed point sets are specified Z-sets? Using known factors of Q, for example Q represented as a product of dendrons, one can construct many involutions of Q with a single fixed point which are not obviously equivalent.

We remark that it is known that the fixed point set of an involution of Q must have trivial homology (this and more general results follow from Smith theory). If the space on page 124 of Theory of Retracts is multiplied by Q, then we obtain very easily an involution of Q whose fixed point set is a Z-set which is a non-simply connected ANR.

Q13. If H, K and $H \cap K$ are all homeomorphic to Q, is $H\ K \cong Q$?

If, additionally, $H \cap K$ is a Z-set in either H or K, then West has shown that $(H \cup K) \times Q \cong Q$. The example of Q12 shows that if H, K, and $H \cup K$ are all homeomorphic to Q, then $H \cap K$ need not be homeomorphic to Q.

Q14. Give necessary and sufficient conditions (or at least reasonable sufficient conditions) such that a map f of Q onto itself be a near homeomorphism.

Curtis has shown that if $f\colon |K| \to |L|$ is a linear surjection, where K and L are contractible finite complexes, and $f^{-1}(|C|)$ is contractible, for each contractible subcomplex of every subdivision of L, then $f \times \mathrm{id}\colon |K| \times Q \to |L| \times Q$ is a near homeomorphism.

QM. Problems about Q-manifolds (See also Q and PF)

A map $f: X \to Y$ is said to be <u>proper</u> provided that the inverse images of compact sets in Y are compact. A homotopy $F: X \times I \to Y$ is said to be proper provided that the entire map F is proper. Then two spaces are said to have the same <u>proper homotopy type</u> provided that the maps and homotopies involved in the usual definition are proper.

There are a number of recent partial results of Chapman concerning Q-manifolds which are $[0,1)$-stable (X is <u>$[0,1)$-stable</u> iff $X \cong X \times [0,1)$), but as indicated below important questions are not yet settled.

The three most important questions in this area appear to be QM1, QM2, and QM3. Because of the local compactness of Q-manifolds, proper maps and proper homotopies seem to provide a convenient framework for studying Q-manifolds.

<u>QM1</u>. What is a topological classification of Q-manifolds, i.e. what are necessary and sufficient conditions for two Q-manifolds X and Y to be homeomorphic?

Chapman has shown that if $X \simeq Y$, then $X \times [0,1) = Y \times [0,1)$, but homotopy type is clearly not strong enough to classify Q-manifolds in general. Soecifically we might ask:

(i) Is it true that Q-manifolds X and Y are homeomorphic iff they have the same proper homotopy type?

(ii) What about (i) in the special case for X and Y assumed to be compact?

Chapman has shown that (i) is true if $X = Q \setminus A$ and $Y = Q \setminus B$, where A and B are Z-sets in Q. In (ii) homotopy type and proper homotopy type coincide, so one might expect compact Q-manifolds to be classifiable according to homotopy type. Chapman has also shown that if X is a compact homotopically trivial Q-manifold, then $X \cong Q$.

<u>QM2</u>. Is there a canonical embedding for Q-manifolds analogous to the open embedding theorem for ℓ_2-manifolds? For example, is it true that each Q-manifold can be embedded in Q as a union of a star-finite collection of basic closed sets in Q?

(A <u>basic closed set in</u> Q is a product of closed subintervals of the interval factors of Q, all but finitely many being the whole factor.) Chapman has shown that if X is any Q-manifold, then $X \times [0,1)$ can be embedded as an open subset of Q. Thus QM2 is true for $[0,1)$-stable Q-manifolds.

<u>QM3</u>. If X is a Q-manifold, then does there exist a locally-compact polyhedron P such that $X \cong P \times Q$?

Chapman has answered this question affirmatively for X an open subset of Q or for X a $[0,1)$-stable Q-manifold. We can weaken QM3 as follows.

<u>QM4</u>. If X is a Q-manifold, then does X have the proper homotopy type of some locally-compact polyhedron?

(i) What about the special case for X assumed to be compact? More generally it is unknown if each locally compact separable metric ANR has the proper homotopy type of a locally compact polyhedron.

<u>QM5</u>. Suppose X and Y are Q-manifolds of the same homotopy type and Y is a compact. Does Y contain a Z-set K such that $X \cong Y \setminus K$?

(i) What about the special case in which $Y = P \times Q$, where P is a compact polyhedron?

(ii) What about the special case in which $Y \cong Q$?

<u>QM6</u>. Let X be a non-compact Q-manifold and let $A \subset X$ be a Z-set. Then does there exist a Z-set $B \subset X$ such that $A \subset B$ and $X \setminus B \cong X$?

This question has an affirmative answer if $X \cong Q \setminus K$, where $K \subset Q$ is a Z-set.

<u>QM7</u>. Let P be a compact polyhedron and let $h: P \times Q \to P \times Q$ be an embedding such that $h(x,0) = (x,0)$, for all $x \in P$, and $h(P \times Q)$ is bicollared. Must the closure of each component of $(P \times Q) \setminus h(P \times Q)$ be homeomorphic to $P \times Q$?

Wong has answered this affirmatively in case P = {point}. A similar result for separable ℓ_2-manifolds is known.

<u>QM8</u>. Let $X \subset Y$, where X is a Q-manifold. Under what conditions is Y a Q-manifold?

West has shown that if $A \times Q \cong B \times Q \cong Q$ and B has Property Z in A, and also $A \setminus B$ is a Q-manifold, then $A \cong Q$. (See also M4).

PF. Problems about Products and Factors (See also Q and QM)

A space Y is a factor of a space X if for some space Z, $Y \times Z \cong X$. By refactoring an infinite product of copies of $Y \times Z$, it is clear that if Y is a factor of Q (or s), then $Y \times Q \cong Q$ (or $Y \times s \cong s$).

West has shown that any contractible finite CW complex is a factor of Q and that a countable infinite product $\Pi_{i=1}^{\infty} Y_i$ of non-trivial factors of Q is homeomorphic to Q. The question naturally arises as to which spaces are factors of Q?

PF1. If X is a compact metric AR, then is $X \times Q \cong Q$? Specifically, is the Borsuk example [Theory of Retracts, 152-156], which is an AR that is not a local AR, a factor of Q? (As an intermediate question one could ask if $X \times I^n$ is a local AR, for some integer $n > 0$, where X is the Borsuk example?)

A metric space X is locally convex equiconnected iff there exists a map $\lambda: X \times X \times I \to X$ such that $\lambda(x_1,x_2,0) = x_1$, $\lambda(x_1,x_2,1) = x_2$, $\lambda(x,x,t) = x$, and such that at every point there is a neighborhood base of λ-convex sets ($A \subset X$ is λ-convex if $\lambda(A \times A \times I) \subset A$).

PF2. Is there any relationship between compact locally convex equiconnected spaces and factors of Q?

We remark that Curtis has shown that every contractible locally finite complex is locally convex equiconnected and every locally convex equiconnected space is locally an AR.

PF3. Is there some way to build up the class of factors of Q (or at least a large subclass of them) from the class of contractible finite complexes, or other known factors?

Along the same lines we could also ask about factors of Q-manifolds.

PF4. If X is a locally-compact separable metric ANR, then is $X \times Q$ a Q-manifold?

West has answered this affirmatively for X a locally-compact CW complex.

For any space X define CX (the <u>cone</u> of X) to be the set $\{v\} \cup X \times [0,1)$, topologized choosing as a basis all open subsets of $X \times [0,1)$ and all sets of the form $\{v\} \cup (X \times (t,1))$, for $t \in (0,1)$. This differs from the usual notion of a cone, but if X is compact the notions are equivalent.

<u>PF5</u>. In the following questions X is a compact metric space.

a. Does $X \times I \cong Q$ imply $X \cong Q$?

b. Does $X \times X \cong Q$ imply $X \cong Q$?

c. Does $X \times \ell^2 \cong \ell^2$ imply $X \times Q \cong Q$?

d. If X is homogeneous and $CX \cong X$, then is $X \cong Q$?

Chapman has shown that $CX \cong Q$ iff $X \times I \cong Q$.

<u>PF6</u>. If X is a topologically complete separable metric AR, then is $X \times s \cong s$?

West has shown this to be true if, e.g., X is a compact CW complex.

<u>PF7</u>. Is every countable infinite product of topologically complete separable metric AR's, with infinitely many non-compact factors, homeomorphic to ℓ^2?

In particular we ask the following question.

<u>PF8</u>. Let m be an infinite cardinal. By an <u>m-spider</u> S^m we mean the "fan" of m half-open unit intervals, joined at a common endpoint, topologized with the "streetcar metric". An <u>m-porcupine</u> P^m is the product of countably many copies of S^m. Is $P^{\aleph_0} \cong \ell^2$? More generally, if m is infinite, then does there exist a Hilbert space H such that $p^m \cong H$?

West has shown that for any infinite m we have $S^m \times H \cong P^m \times H \cong H$, for some Hilbert space H.

<u>PF9</u>. If X is a topologically complete separable metric ANR, then is $X \times \ell^2$ an ℓ^2-manifold?

As in PF4 this is known for X a locally-compact CW complex.

PF10. Let X be a topologically complete separable metric space.

a. Does $X \times (0,1) \cong s$ imply $X \cong s$?

b. Does $X \times Q \cong s$ imply $X \cong s$?

c. Does $X \times X \cong s$ imply $X \cong s$?

d. Does $X \times Y \cong s$ imply $X \cong s$, where Y is some factor of Q?

e. If X is homogeneous, non-compact, and $X \cong CX$, then is $X \cong s$?

M. Problems about E-manifolds

M1. If M is a compact n-manifold, is the space H(M) of homeomorphisms of M onto M an ℓ_2-manifold (under the sup norm topology)? Or, weaker, is H(M) an ANR?

(i) What about the special case in which M is an n-sphere?

(ii) Is the space of homeomorphisms on an n-cell, which are fixed on the boundary, homeomorphic to ℓ_2?

Anderson has affirmatively answered (i) and (ii) for n = 1. In general M1, (i), or (ii) are important questions and any reasonable partial results would be worthwhile. Ross Geoghegan has recently proved that for some general classes of function spaces H which include M1 as a special case, $H \cong H \times \ell_2$. J.E. Keesling has extended Geoghegan's results to include, for example, the space of homeomorphisms of a metric space which admits a non-trivial continuous one-parameter flow. It is known that if $H \cong H \times \ell_2$, then any σ-compact subset of H is negligible. W.K. Mason has shown that the space of homeomorphisms of the 2-cell fixed on the boundary is an absolute retract. This result may make (ii) above accessible for the case n = 2.

M2. The Eells-McAlpin theorem (which uses a version of Sard's theorem) shows that given any closed set $K \subset M$ (where M is an ℓ_2-manifold) and any neighborhood V of K, there is a closed neighborhood U such that $U \subset V$, $(U, \partial V)$ is a manifold with boundary, and $K \subset U \setminus \partial U$. An easy topological proof using s can be given, but a topological proof valid in the non-separable case would be interesting.

M3. For M a separable F-manifold, can every homeomorphism of M onto itself be approximated by diffeomorphisms?

Burghelea and Henderson have proved that such homeomorphisms are isotopic to diffeomorphisms.

M4. Let X be a topologically complete separable metric space.

(i) If X is an ANR, $Y \subset X$ is dense in X, and Y is a separable F-manifold, under what conditions can we conclude that X is a separable F-manifold?

(ii) If X is an ANR, Y is an F-manifold, and Y is open and dense in X, under what conditions can we conclude that X is a separable F-manifold?

(iii) Let M be a separable F-manifold, and suppose that $X \subset M$ is the closure of an open set Y. Under what conditions can we conclude that X is a separable F-manifold?

Henderson has observed relative to (i), for example, that if Z-sets are strongly negligible in X and if $X \setminus Y$ is a countable union of Z-sets, then $X \cong Y$. However, it seems difficult to verify these conditions in many naturally arising cases.

<u>M5</u>. Let M be an E-manifold and let A be an open subset of M. If $h: A \to E$ is an open embedding which admits an extension to Cl(A), under what conditions can h be extended to an open embedding of M into E? If, for some neighborhood U of Cl(A), h can be extended to an open embedding of U, can h be extended to an open embedding of M into E?

These questions are open for $E = \ell^2$.

In the following three problems we assume K and M to be E-manifolds and K to be a closed subset of M. Then K is said to have <u>local deficiency</u> n <u>at a point</u> p if there exist an open set U with $p \in U$ and a homeomorphism h of $(-1,1)^n \times E$ onto U such that h $h(\{0\}\times E) = K \cap U$. If K has local deficiency n at every point of K then we say that K has local deficiency n. Let $R \subset K$ such that (a) R consists of a single point, (b) R is compact or (c) R is a Z-set in M and a Z-set in K. The three problems below are not known for $E \cong \ell_2$.

<u>M6</u>. If K has local deficiency 1 at every point of $K \setminus R$, does K have local deficiency 1 for cases (a), (b) and (c) above.

<u>M7</u>. For $n > 1$, under what conditions does local deficiency n at every point of $K \setminus R$ imply that K has local deficiency n for cases (a), (b) and (c) above? Kuiper has given examples for $n = 2$ where R is a single point, an arbitrary n-cell, or a copy of E, such that K does not have local deficiency 2. The examples involve knots. For $n > 2$ no examples are known.

M8. For $n > 1$, does local deficiency n imply the existence of a neighborhood U of K such that U is the total space of a fibre bundle over K with fibre $(-1,1)^n$?

M9. Let M and K be E-manifolds with $K \subset M$ and K a Z-set in M. Then K may be considered as a "boundary" of M, i.e. for any $p \in K$ there exists an open set U in M and a homeomorphism h of U onto $E \times (0,1]$ such that $h(K \cap U) = E \times \{1\}$.

Under what conditions on the pair (M,K) does there exists a homeomorphism h of M into E such that the topological boundary of $h(M)$ in E is $h(K)$? This is not known in general even for ℓ_2. It is known that if the identity map of K into M induces a homotopy equivalence of K and M, then the embedding is possible.

M10. Let A be a closed subset of the E-manifold M such that for each closed $B \subset A$, $M \setminus B \cong M$. Must A be a Z-set in M? This is not known even for $M = \ell_2$.

M11. Let $\xi: E \to B$ be a fibre bundle over a paracompact space B with fibre F an ℓ_2-manifold. Suppose K is a closed subset of E such that $K \cap \xi^{-1}(b)$ is a Z-set in each $\xi^{-1}(b)$. Is there a fibre-preserving homeomorphism of $E \setminus K$ onto E?

LG. Problems about Linear Groups

A function $T: E \to F$ of the Banach spaces E and F is linear if $T(x+y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$ for all $x,y \in E$ and $\lambda \in R$. Let $L(E,F)$ be the space of all continuous linear functions $T: E \to F$ with the norm topology ($|T| = \sup\{|T(x)|_F: |x|_E \leq 1\}$) and let $L(E) = L(E,E)$. An operator $T \in L(E)$ is invertible if it has a continuous inverse. It is known that each 1-1 and onto operator is invertible. The general linear group of E, denoted $GL(E)$, is the group of invertible operators of E with the norm topology where the group operation is composition of functions. If E is a Hilbert space with inner product $(|)$ and $T \in L(E)$, then there exists a unique $T \in L(E)$ such that $(T(x)|y) = (x|T^*(y))$ for all $x, y \in E$. T^* is called the adjoint of T and T is said to be unitary if $T^*T = TT^* = I$. It is known that T is unitary iff T is isometric ($|T(x)| = |x|$ for all $x \in E$) and onto.

LG1. Is there a contraction $h_t: G \to G$ with $h_0 = \text{id}$ and $h_1(g) = 1$ for all $g \in G$ such that for each $t \in I$, $h_t(g_2\, g_1) = h_t(g_2) \circ h_t(g_1)$ where

a) G = Group of homeomorphisms of the unit sphere $S(\ell_2)$ in separable Hilbert space with the norm topology?

b) G = Group of diffeomorphisms ... ?

(Note: No, by Kuiper, for $G = GL(\ell_2)$ with norm topology and for G = Unitary group of ℓ_2 with norm topology. Yes, by Renz, for $G = \text{Homeo}(\ell_2)$ with compact open topology.)

LG2. Is there a Banach space B with two non-equivalent complex (algebraic) structures? That is, do there exist automorphisms $\sigma_j: B \to B$ ($j = 1,2$) such that $\sigma_j^2 = -\text{id}$ and for no automorphism τ do we have $\tau^{-1}\sigma_2\tau = \sigma_1$? Prove that $c_0 \oplus \ell_2$ is not a counterexample.

LG3. Let $0 \to R^m \to B \to B_n \to 0$ be a split embedding of R^n in B with complement B_n. Thus $B = R_n \oplus B_n$. Let $GL(n) = GL(R^n)$ (= all $n \times n$ invertible matrices) and define the embedding $i: GL(n) \to GL(B)$ by $i(g) = g \times \text{id}_{B_n}$. Is i homotopic to a constant map in $GL(B)$ for all n and all Banach spaces B? (If true for B then it seems that Fredholm manifolds modelled on B are B-parallelizable.)

LG4. Equivariant contraction of GL(H). Let G be a subgroup of GL(E), $E = \ell_2$ or $\mathbb{R}^n$, and consider the action on $\ell_2(E) = \{x=(x_1,x_2,\dots): x_i \in E$ with norm $|x|^2 = \sum |x_i|^2 < \infty\}$ given by $gx = (gx_1, gx_2, \dots)$ for $g \in G$. Is there a contraction ϕ_t: $GL(\ell_2(E)) \to GL(\ell_2(E))$ such that $\phi_0 = \mathrm{id}$, $\phi_1(GL(\ell_2(E)) = e$, and $g\phi_t(h)g^{-1} = \phi_t(ghg^{-1})$ for all $g \in G$, $h \in GL(\ell_2(E))$, and $t \in I$? This is true for G a compact Lie group (Segal: Bull. London Math. Soc. 1 (1969), 329-331). Is it true for g generated by $\binom{11}{01}$ acting on $E = \mathbb{R}^2$?

LG5. There are only very few homotopy types that are known to be homotopy types of GL(B) for some Banach space B. Find necessary conditions for such homotopy types. Find Banach spaces with new homotopy types.

LG6. Let $L^2(H,H)$ be the group of 2-jets of diffeomorphisms ϕ: $(H,0) \to (H,0)$ at 0. There is a natural projection $L^2(H,H) \to GL(H)$. Are there interesting subgroups that project onto GL(H) or onto the Fredholm operators in GL(H)?

D. Problems about Differential Topology

Historically, the study of infinite dimensional manifolds was first done in the differential category. James Eells, Jr. gives a survey of results in "A Setting for Global Analysis", Bull. A.M.S. 72 (1966), 751-807. Some recent results (1968) that have been very influencial in the area are the following theorems which are the combined efforts of Eells-Elworthy, Kuiper-Burghelea, and Moulis.

Let M,N be separable C^∞-Hilbert manifolds.
Then (1) M is C^∞-diffeomorphic to an open subset of ℓ_2,
(2) M is C^∞-diffeomorphic to $M \times \ell_2$,
and (3) M and N are C^∞-diffeomorphic iff they are of the same homotopy type.

<u>D1</u>. It is known that $\ell_2 = H$ and $S = \{x \in H: ||x||=1\}$ are C^∞-equivalent. Is there a real analytic equivalence? Remark: H and $H\setminus\{0\}$ are real analytically equivalent. [Burghelea and Kuiper, Annals of Math.] Is every homeomorphism on an open set in H homotopic to a real analytic diffeomorphism? (By Burghelea and Henderson, if it is homotopic, then it is also isotopic.) Let X and Y be open diffeomorphic subsets of H. Are they real analytically equivalent? Can every separable paracompact real analytic H-manifold be analytically embedded as a closed subset of H.

<u>D2</u>. A. Douady has shown the existence of a C^∞-diffeomorphism from ℓ_2 onto $B^0(1)\subset\ell_2$ of the form $1+\alpha$ where α is locally finite-dimensional (i.e. each point has a neighborhood on which α has range in a finite dimensional subspace of ℓ_2). For L_1 no such C^1-differentiable mapping exists. For which Banach spaces E does there exist a diffeomorphism $E \to B^0(1) \subset E$ of this form?

<u>D3</u>. Let M, X be separable C^r ℓ_2-manifolds and let $f: M \to X$ be a C^r-map with the property that for each $x \in X$ there is a neighborhood U of x and open embeddings $f^{-1}(U) \to \ell_2 \times \ell_2$ and $U \to \ell_2$ where the following diagram commutes.

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[d] & U \arrow[d] \\ \ell_2 \times \ell_2 \arrow[r, "p_2"] & \ell_2 \end{tikzcd}$$

Are there open embeddings $M \to \ell_2 \times \ell_2$ and $U \to \ell_2$ where the following diagram commutes?

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & U \arrow[d] \\ \ell_2 \times \ell_2 \arrow[r, "p_2"] & \ell_2 \end{tikzcd}$$

D4. The Banach spaces c_0 and H have C^∞-norms. With some care we can find a C^∞-norm for $c_0 \oplus H$ [Kuiper-Terpstra: Differentiable closed embedding of Banach manifolds]. Find a C^∞-norm for

$$H(c_0) = \{(x_i): x_i \in c_0 \quad \text{and} \quad \textstyle\sum_1^\infty |x_i|^2 < \infty\}$$

with norm $||x|| = \sqrt{\sum_1^\infty |x_i^2|}$ and for

$$c_0(H) = \{(x_i): x_i \in H \quad \text{and} \quad \lim_{i\to\infty} |x_i| = 0\}$$

with norm $||x|| = \sup |x_i|$.

D5. Let B be a Banach space with C^q-norm. If the B-manifolds X and Y are tangentially homotopy equivalent, are they C^q-diffeomorphic? Special cases: (a) $q = \infty$, (b) GL(B) is contractible (e.g. $B = \ell_p$).

D6. Is there for any pair of open sets U and V in a non-separable Hilbert space H, such that $U \subset \bar{U} \subset V \subset H$, a C^∞-function ϕ on H with $\phi(x) = 0$ for $x \in U$ and $\phi(x) = 1$ for $x \notin V$?

D7. If there is a non-zero C^k function on a separable Banach space B with bounded support, is there a C^k norm?

D8. Can one find a diffeomorphism $H \times H \to H$ as close as we like to the projection onto the second coordinate?

D9. Let X be a C^∞ H-manifold, homotopically equivalent to a finite CW complex with n cells. Is there a closed embedding f of X in H such that for "almost all" linear functionals $\xi: H \to R$, $\xi \circ f$ is non-degenerate with $\leq 2n$ critical points?

D10. Let $E_i \to X$ $(i = 1,2)$ be differentiable fiber bundles over X where X is a C^∞-paracompact separable manifold and E_i is a C^∞ paracompact separable H-manifold. D. Burghelea says he has shown that if X is a compact manifold, then fiber homotopy equivalence implies the bundles are isomorphic. Is this true for other finite or infinite-dimensional Hilbert manifolds?

D11. Is there a reflexive Banach space B and a B-manifold X which cannot be split C^1-embedded in B [or in $B \oplus B$] ? Is there a reflexive Banach space B which is not isomorphic with $B \oplus B$? For the Banach space J of R.C. James, $J \oplus J$ does not embed linearly in J, but J is not reflexive.

For the following three problems let M be a complete Riemannian C^∞-manifold modelled on separable Hilbert space H.

D12. If $p,q \in M$, is it possible to join p and q by a goedesic? Answer is yes if $\dim H < \infty$ or if p and q are close to each other.

D13. Do there exist (necessary and?) sufficient conditions on M so that two points $p,q \in M$ can be joined by a geodesic whose length is exactly the Riemannian distance $\delta(p,q)$? Answer is no without extra conditions on M (McAlpin "ellipsoid" counterexample).

D14. Let (M_i) be a sequence of Hilbert manifolds. Is there a "product" $M = "\Pi" M_i$ such that the tangent space at p is a Hilbert direct sum of convenient tangent spaces to the M_i's? If the answer is no, there still might be a "product" of manifolds with base point (M_i, p_i) [e.g. true if for each i, $(M_i, p_i) = (H,0)$].